

A general approach to the localization of unstable periodic orbits in chaotic dynamical systems

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We present a method to detect the unstable periodic orbits of a multidimensional chaotic dynamical system. Our approach allows us to locate in an efficient way the unstable cycles of, in principle, arbitrary length with a high accuracy. Based on a universal set of linear transformations the originally unstable periodic orbits are transformed into stable ones and can consequently be detected and analyzed easily. This method is applicable to dynamical systems of any dimension and requires no preknowledge with respect to the solutions of the original chaotic system. As an example of application of our method we investigate the Ikeda attractor in some detail.

I. INTRODUCTION

The finding that the unstable periodic orbits provide a skeleton for the organization of the very complex chaotic dynamics can be considered as one of the major advances in our understanding of the behaviour of nonlinear dynamical systems during the past ten years. Many physical quantities of these systems can be determined by knowing the positions/properties of the unstable periodic orbits living in the chaotic sea. For strange attractors belonging to dissipative systems the quantitative description of their structural properties in terms of unstable periodic orbits has been successfully used for several dynamical systems [1–5]. This reflects the importance of the cycles for the analysis and decoding of the dynamics on the attractor. On the other hand knowing the positions of the cycles in phase space one can use them to control chaotic dynamical systems [6]. Moreover the quantum mechanical properties of classically chaotic conservative systems possess in the semiclassical regime a series expansion with respect to the lengths and the stability coefficients of the periodic orbits (see ref. [7] and references therein). Since chaotic behaviour is an intrinsic property of many dynamical systems periodic orbit theory possesses numerous applications in different areas of physics.

Assuming that the cycles represent the skeleton for the chaotic dynamical evolution of dissipative systems one can use an appropriate series expansion in terms of their period to determine observables like fractal dimensions, average Lyapunov exponents, entropies or the invariant measure of the corresponding strange attractor [1–3,8,9]. The convergence properties of this expansion allow us to test the applicability of current periodic orbit theory for the properties of dissipative dynamical systems. The most important and at the same time most difficult step of this procedure is to find the location of the unstable cycles in the chaotic sea: in spite of the fact that their number is growing exponentially with

increasing period and that they are dense in the chaotic sea the unstable cycles are of measure zero whereas the chaotic orbits are (for a fully chaotic system) of measure one. However, once the location of the orbits is known it is straightforward to extract their properties from the underlying equations of motion. It is therefore not surprising that a number of recent works deal with the development of efficient methods and strategies for the detection of the periodic orbits [10–12].

The application of the Newton-Raphson method to find the unstable cycles requires a good guess for the starting point of the iterative procedure. For cycles of higher period this is only possible if one uses a very fine grid of initial conditions on the attractor. Consequently this approach involves a huge numerical effort for more than two dimensions and is therefore in general not feasible. Alternative methods can be found in the literature but their applicability is limited to special low-dimensional systems [10,12,13].

The purpose of the present investigation is to establish a generally applicable as well as reliable and accurate method in order to determine the unstable periodic orbits in a chaotic dynamical system [14]. The basic idea of our approach is to use an universal set of linear transformations in order to transform the unstable periodic orbits of the original system into stable periodic orbits which occupy the same positions in coordinate space. The stable periodic orbits of the transformed systems can then be found by simply iterating these dissipative transformed systems. It can be shown that such transformations exist in general and possess very restrictive and simple forms as well as an appealing geometrical interpretation in terms of a vector field which is organized through the positions of the cycles. To demonstrate the efficiency of our method we apply it to the attractor of the two dimensional Ikeda map [15] which describes the dynamics of an optical field in a ring cavity. We locate the unstable periodic orbits up to period 13 of the Ikeda strange attractor. Subsequently we derive their stability coefficients and use the expansions according to periodic orbit theory in order to determine the average Lyapunov exponents, the fractal dimension as well as the topological entropy of the Ikeda attractor.

The paper is organized as follows : In section 2 we present the general theoretical framework of our method and discuss in more detail the case of a two dimensional system. Furthermore we present a universal algorithm for determining the unstable periodic orbits of a multidimensional chaotic dynamical system. In section 3 we use this algorithm to locate the unstable periodic orbits of the Ikeda attractor. From their location we derive the corresponding stability coefficients which can be used to determine the average Lyapunov exponents, the fractal dimension and the topological entropy of the attractor. In section 4 we summarize the main aspects of our approach and briefly report on possible applications as well as perspectives related to the general problem of solving nonlinear equations.

II. THE THEORETICAL FRAMEWORK

Let us consider a N -dimensional discrete fully chaotic dynamical system given by:

$$U : \quad \vec{r}_{i+1} = \vec{f}(\vec{r}_i) \quad (1)$$

U , being fully chaotic, possesses only unstable fixed points (FP). Since points of a periodic orbit of period p are FP of the p -th iterate $\vec{f}^{(p)}$ the term FP stands in the following for periodic orbits in general, i.e. for orbits of any period. We thereby have to replace \vec{f} in eq.(1) by $\vec{f}^{(p)}$. Our goal is to construct from the map U in eq.(1) different dynamical systems $\{S_k | k = 1, \dots, M\}$ (see below) which possess FP at the same positions as U but instead of being unstable they have become stable

in the dynamical systems $\{S_k\}$. The corresponding transformations $\{L_k : U \rightarrow S_k\}$ are required to preserve the number of FP, i.e. no additional orbits should be created. The transformations $\{L_k\}$ therefore change the stability properties but not the location of the FP. The index k stands for the possibility that different orbits may require different transformations for their stabilization. However, as we shall see below, each type of transformation will stabilize not only a single FP but a whole rather general class of infinitely many FP ranging up to arbitrarily high periods. If we succeed with our plan then the search for the positions of the FP of the system U becomes a manageable task: because of the stability of the FP in the transformed dissipative systems $\{S_k\}$ each trajectory of S_k after some iterations closely approaches a FP \vec{r}_F . Per construction \vec{r}_F is then also a FP of the system U and we therefore know the position of the FP in the original system U .

To fulfill the requirement of the one to one correspondence between the FP of U and S_k the transformation L_k should in general be linear. Consequently S_k takes on the following appearance:

$$S_k : \quad \vec{r}_{i+1} = \vec{r}_i + \mathbf{\Lambda}_k(\vec{f}(\vec{r}_i) - \vec{r}_i) \quad (2)$$

where $\mathbf{\Lambda}_k$ is a constant invertible $N \times N$ matrix. The definition (2) satisfies the one to one correspondence of the FP of U and those of S_k : If $\vec{r}_i = \vec{r}_F$ is a FP of U then the parenthesis on the right hand side of eq.(2) vanish and therefore \vec{r}_F is also a FP of S_k . On the other hand if \vec{r}_F is a FP of S_k and since $\mathbf{\Lambda}_k$ is nonsingular the parenthesis on the right hand side of eq.(2) must be equal to zero for $\vec{r}_i = \vec{r}_F$ which implies that \vec{r}_F is also a FP of U . Thus the dynamical laws U and S_k possess FP at identical positions in space.

Our next step is to stabilize the FP of the transformed systems S_k by suitable choices for $\mathbf{\Lambda}_k$. Different unstable FP of the map U are then, in general, stable in different transformed dynamical laws S_k . However, as we shall see below, the number of matrices $\{\mathbf{\Lambda}_k\}$ necessary to achieve stabilization of *all* FP is very small. It turns out that if the absolute values of the elements of the matrices $\mathbf{\Lambda}_k$ are sufficiently small ($|\lambda_{ij}^{(k)}| \ll 1$, $i, j = 1, \dots, N$) then there exists a universal set of very restrictive matrices such that at least one matrix $\mathbf{\Lambda}_k$ belonging to this set transforms (via eq.(2)) a given unstable FP of U to a stable FP of the corresponding map S_k . In order to determine this set of matrices let us consider the stability matrices \mathbf{T}_U and \mathbf{T}_{S_k} of U and S_k which obey the following relation:

$$M_k : \quad \mathbf{T}_{S_k} = \mathbf{1} + \mathbf{\Lambda}_k(\mathbf{T}_U - \mathbf{1}) \quad (3)$$

For \mathbf{T}_U we assume that it is real, invertible and diagonalizable. Since \vec{r}_F is an unstable FP at least one of the eigenvalues of \mathbf{T}_U at \vec{r}_F must possess an absolute value greater than one. In order to stabilize \vec{r}_F we proceed in two steps: first we use the parametrization $(\mathbf{\Lambda}_k)_{ij} = (\lambda \mathbf{C}_k)_{ij}$ with $1 \gg \lambda > 0$ and $C_{ij} = O(1)$. The matrix C_{ij} has to be chosen such that the real parts of all eigenvalues of the matrix $\mathbf{C}_k \cdot (\mathbf{T}_U - \mathbf{1})$ are negative. If this is achieved then the next step is to use a sufficiently small value for the parameter λ , such that the eigenvalues of the matrix $\mathbf{T}_{S_k} = \mathbf{1} + \lambda \mathbf{C}_k(\mathbf{T}_U - \mathbf{1})$ have absolute values less than one. It can be shown that this is always possible if λ is sufficiently small: for sufficiently small values of λ the leading contribution with respect to λ in the absolute values of the, in general complex, eigenvalues is of order λ^1 and emerges from the real part of the eigenvalues of the transformed system. Consequently we have transformed the unstable FP \vec{r}_F of U via L_k into a stable FP of S_k thereby keeping its position fixed.

Next let us derive possible sets of matrices $\{\mathbf{C}_k\}$ which, according to the above discussion, transform the signs of the real part of the eigenvalues of $\mathbf{T}_U - \mathbf{1}$ and consequently allow us to stabilize any configuration of unstable FP. First we provide a rather general set of matrices and subsequently

we will specialize to a much more restrictive and simpler set. For any given T_U it is possible to find an involutory matrix C_k defined through $C_k^2 = \mathbf{1}$ such that $A = C_k(T_U - \mathbf{1})$ has eigenvalues with negative real parts. The proof reads as follows: Assuming the diagonalizability of the matrix $B = T_U - \mathbf{1}$ there exists a similarity transformation P such that $B_D = P^{-1}BP$ is diagonal. We have then:

$$P^{-1}AP = P^{-1}C_kP \cdot B_D$$

We can therefore always choose a diagonal matrix $C_{Dk} = P^{-1}C_kP$ with elements ± 1 on its diagonal such that $C_{Dk} \cdot B_D$ has eigenvalues with negative real part. Such a matrix C_{Dk} is according to its definition involutory. Due to the invariance of the eigenvalue spectrum of a matrix with respect to similarity transformations, the matrix A has then also eigenvalues with negative real part. In addition if C_{Dk} is involutory then any matrix resulting from C_{Dk} through similarity transformations, and in particular C_k , is also involutory. We can therefore always find an involutory matrix C_k given by $C_k = PC_{Dk}P^{-1}$ such that $C_k(T_U - \mathbf{1})$ possesses eigenvalues with negative real parts.

The set of involutory matrices is for our purposes rather general. For reasons of applicability as well as practical efficiency of our method it is therefore desirable to find a smaller set of transformation matrices. In fact it turns out that a much more restricted form for the matrices C_k is sufficient in order to achieve stabilization of any FP via the transformations $\{L_k\}$, namely all the matrices corresponding to special reflections and rotations in space. The elements of this new class of matrices $\{C_k\}$ are $C_{ij}^{(k)} \in \{0, \pm 1\}$ and each row or column contains only one element which is different from zero. The matrices C_k are therefore orthogonal. The number a_N of such matrices in N dimensional space is given by $a_N = N! 2^N$.

To illuminate our approach we discuss in the following the case $N = 2$ in more detail. In this case U is given by:

$$\begin{aligned} x_{i+1} &= f(x_i, y_i) \\ y_{i+1} &= g(x_i, y_i) \end{aligned} \tag{4}$$

and represents a fully chaotic 2-D map with a 2×2 stability matrix T_U . Let us denote with $\rho_{1,2}$ the eigenvalues of T_U . Being fully chaotic U possesses only hyperbolic FP. The choice of the matrix C_k appropriate for the stabilization of a particular FP depends on the class to which the hyperbolic FP belongs. According to the above discussion there are $a_2 = 8$ possible matrices $\{C_k\}$ as candidates to be used for this stabilization process. With some algebra one can find the matrices which achieve the stabilization of the different types of unstable FP:

- For hyperbolic FP with reflection and $\rho_1 < -1, -1 < \rho_2 < 0 \Rightarrow C_1 = \mathbf{1}$
- For hyperbolic FP with reflection and $\rho_1 < -1, 0 < \rho_2 < 1 \Rightarrow C_1 = \mathbf{1}$
- For hyperbolic FP with reflection and $\rho_1 > 1, -1 < \rho_2 < 0$ or hyperbolic FP without reflection ($\rho_1 > 1, 0 < \rho_2 < 1$) we have to distinguish between the following three cases:
 - (a) $\frac{\partial f}{\partial x}|_{FP} > \frac{\partial g}{\partial y}|_{FP} \Rightarrow C_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
 - (b) $\frac{\partial f}{\partial x}|_{FP} < \frac{\partial g}{\partial y}|_{FP} \Rightarrow C_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
 - (c) $\frac{\partial f}{\partial x}|_{FP} = \frac{\partial g}{\partial y}|_{FP} \Rightarrow C_4 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{or} \quad C_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

With the above choices for \mathbf{C}_k and sufficiently small values of the parameter λ we can stabilize any hyperbolic FP of a chaotic 2-D system. The only exception where our approach does not work is the case of a parabolic FP, i.e. for parallel flow ($\rho_1 = 1 \vee \rho_2 = 1$). However this case is of no interest to the present investigation.

It is a major advantage of the present approach that a single matrix $\{\mathbf{C}_i\}$ is responsible for the stabilization of an infinite number of unstable FP belonging to periodic orbits of arbitrarily high periods. For example: the matrix \mathbf{C}_1 stabilizes hyperbolic FP (periodic orbits) with reflection with respect to both invariant manifolds, independent of their period. We remark that only 5 of the 8 matrices \mathbf{C}_k are necessary to stabilize all hyperbolic FP in two dimensions. The matrices \mathbf{C}_4 and \mathbf{C}_5 typically stabilize no or only a very few additional FP which is due to the fact that the condition $\frac{\partial f}{\partial x}|_{FP} = \frac{\partial g}{\partial y}|_{FP}$ is met only very rarely. A similar statement holds also for higher dimensional systems: only a subset of the above given class of a_N matrices is actually necessary in order to achieve the stabilization of any given FP. According to the above we have shown a one way criterium with respect to the stabilization of the FP. For example: any hyperbolic FP with reflection ($\rho_1 < -1, -1 < \rho_2 < 0$) in two dimensions can in particular be stabilized by using the matrix \mathbf{C}_1 but might also be stabilized by using other matrices of the set \mathbf{C}_k . The explicit statements saying which matrices stabilize which types of FP become increasingly more complex with increasing dimension of the dynamical system under investigation.

Having presented the basic features of our method we discuss now briefly how it can be used to detect the unstable FP of a given N -dimensional discrete fully chaotic dynamical system. Using eq.(2) we transform the given dynamical system U into a new system S_k . For the matrix $\mathbf{\Lambda}_k = \lambda \mathbf{C}_k$ we use a sufficiently small value of λ and \mathbf{C}_1 from $\{\mathbf{C}_k\}$, $k = 1, \dots, a_N$. Iterating an arbitrary initial point \vec{r}_0 with S_1 from eq.(2) forward in time we observe the following characteristic behaviour: either the trajectory runs to a stabilized FP \vec{r}_F with steps of continuously decreasing size or, if stabilization is not achieved using \mathbf{C}_1 , the trajectory chaotically evolves on the attractor or escapes to infinity. In order to get all the FP of U this procedure has to be repeated for a representative set of starting points which covers in a crude way (see below) the phase space of the system. Subsequently we perform the same procedure for the next matrix \mathbf{C}_2 of the set \mathbf{C}_k , etc... until all the matrices $\{\mathbf{C}_k\}$ have been used. Following this procedure we obtain the complete set of FP of U . In order to find the unstable FP of period i we simply have to replace \vec{f} in eq.(2) by its i -th iterate \vec{f}^i . One must however be careful with respect to the choice of the value of λ in $\mathbf{\Lambda}_k$ because it is indirectly related to the stability coefficients of the desired cycle. We therefore have to use decreasing values of λ with increasing periods of the cycles we would like to detect. However later on in the present section we will provide a continuous formulation of the transformed systems which is independent of the value of the parameter λ . Due to the fact that different kinds of FP (e.g. hyperbolic FP with and different kinds without reflection) are stabilized by different matrices \mathbf{C}_k our stabilization procedure offers the possibility to distinguish between the different types of FP.

An important advantage of our stabilization method is its global character. Even points lying far from the linear neighbourhood of the stabilized FP are attracted to it after a few iterations of the transformed dynamical law. To each FP of a certain period we can assign its basin of attraction consisting of the set of starting points which converge towards the stabilized FP if we iterate them with the transformed dynamical system S_k . With increasing period the typical volume of a basin of attraction becomes increasingly smaller and the basins form a geometrically very complex and

interwoven network which covers the attractor.

Using the above approach to detect the unstable periodic orbits of a given chaotic dynamical system it turns out that the number of starting points needed to obtain the periodic orbits of a given period on the (closure of the) attractor is only a few times more than the number of cycle points themselves. Suitable starting points for the application of the transformed dynamical law S_k can be obtained by using, for example, a chaotic trajectory on the attractor itself or more refined methods which cover the attractor in a more systematic manner. We used a simple empirical strategy in order to seek for the periodic orbits of a given period. Assuming that a certain number of periodic orbits of period p have been found we iterate a multiple of the starting points used in the previous run. If no additional periodic orbits show up we take the set of detected orbits as complete. We emphasize that this is by no means a proof for the completeness of the set of detected periodic orbits. However it turns out that for dynamical systems for which the periodic orbits are known, like for example the Henon map, the above procedure yields all periodic orbits. Because of the fact that each FP possesses a finite basin of attraction (see below and Fig.2) the above method turns out to be rather insensitive with respect to the specific choice of the set of starting points covering the attractor in a crude and large-meshed manner.

The transformed dynamical law (2) possesses an appealing geometrical interpretation. Let $\{\vec{r}_j, j = 1, \dots, p\}$ be a trajectory of the original system U . At each point of the trajectory we define a vector field $\vec{V}_U(\vec{r}_j) = \vec{r}_{j+1} - \vec{r}_j$. The corresponding transformation L_k represents then a special reflection/rotation of each vector $\vec{V}_U(\vec{r}_j)$ combined with a subsequent scale transformation of its length with the factor λ . Using eqs.(1,2) we obtain for the vector field \vec{V}_{S_k} of the transformed map

$$\vec{V}_{S_k}(\vec{r}) = \mathbf{\Lambda}_k \vec{V}_U(\vec{r}) \quad (5)$$

The main feature of the new vector field \vec{V}_{S_k} is its global organization around those FP which have been stabilized. The flow of the vector field \vec{V}_{S_k} is centered and organized around the positions of the stable FP which represent sinks/sources of this vector field. To illustrate these properties we show in Fig.1 selected vector fields belonging to a chaotic trajectory on the attractor of the Ikeda map (see also section 3). In Fig.1(a) the vector field \vec{V}_{S_k} for the first iterate \vec{f}^1 is illustrated. We use a trajectory of 200 points and the stabilization matrix $\mathbf{\Lambda}_k = 0.1 \cdot \mathbf{C}_1$. The global organization of the flow towards the FP $\vec{r}_F = (0.5328, 0.2469)$ whose position is indicated in Fig.1 by a cross is evident. Obviously this property is not restricted or specific for the linear neighbourhood of the FP but represents a global feature of the dynamical system. Starting with any point on the attractor the trajectory of the transformed map moves immediately towards the FP and yields with increasing number of iterations increasingly accurate values for its position. In Fig.1(b,c) we show the vector field \vec{V}_{S_k} in the vicinity of the FP $\vec{r}_{F,1} = (0.5098, -0.6084)$ and $\vec{r}_{F,2} = (0.6216, 0.6059)$ for the second iterate $\vec{f}^{(2)}$ of the Ikeda map which represent a period two cycle of the map. In Fig.1(b) 200 points in the neighbourhood of $\vec{r}_{F,1}$ are plotted. The stabilization matrix $\mathbf{\Lambda}_k = 0.02 \mathbf{C}_1$ was used. In Fig.1(c) we use the same number of points as well as the same stabilization matrix $\mathbf{\Lambda}_k$ to illustrate the organization of the vector field around $\vec{r}_{F,2}$. Again the positions of the FP are indicated by crosses. From Figs.1(b,c) it is evident that the flow of the vector field \vec{V}_{S_k} can exhibit sharp, but still smooth, turns in the immediate neighbourhood of the FP. The size of these regions of nonlinearity in general decreases with increasing period of the cycle considered.

The fact that the stabilized FP represent the centers of the flow of the vector field \vec{V}_{S_k} possesses a counterpart in the original chaotic system U which we shall discuss briefly in the following. If a

chaotic trajectory approaches a FP or more precisely enters the linear neighbourhood of the FP it can be shown that the maximum of the deflection of the trajectory in the linear regime occurs for the position closest to the FP. The deflection of the incident trajectory can be considered as a turning of the trajectory and the point of maximum deflection (or curvature for continuous systems) can be considered as a generalization of the notion of a turning point in one dimension (see refs. [16]). Different chaotic trajectories approaching the FP from different directions in space are deflected to different directions of their outgoing manifold. The common feature of chaotic trajectories in the neighbourhood of a FP is the fact that the above-mentioned deflection or turning occurs with respect to any direction of space. If the chaotic system U is transformed via the corresponding stabilizing transformation L_k , the resulting dynamical system S_k possesses centers of the corresponding flow of the transformed system at the positions of the FP. L_k can therefore be considered as a transformation from spreading to focusing flow.

The above-described method of detecting the unstable FP (periodic orbits) for a given chaotic dynamical system involves the parameter λ which has to be chosen sufficiently small in order to transform the unstable FP of the original chaotic system to a stable one via the transformations L_k . With increasing period of the FP to be located, the parameter λ has to be chosen increasingly smaller in order to achieve stabilization. However we should not choose λ too small since the convergence of the iterated transformed dynamical laws (see eq.(2)) to the FP will then become very slow and involves a waste of computer time. There is a simple way to avoid this tuning of the parameter λ which makes our approach independent of the parameter λ . Taking the limit $\lambda \rightarrow 0$ in eq.(2) we obtain

$$\lim_{\lambda \rightarrow 0} \frac{(\vec{r}_{i+1} - \vec{r}_i)}{\lambda} = \dot{\vec{r}} = \mathbf{C}_k(\vec{f}(\vec{r}) - \vec{r}) \quad (6)$$

This equation represents the continuous formulation of our transformed discrete dynamical systems in eq.(2). Its solutions possess the same nice properties as those of eq.(2) in the sense that any stabilized trajectory is running to a stable FP. The important difference however is that the solutions of eq.(6) do not depend on the parameter λ and it can therefore easily be solved by using any suitable integration scheme like, for example, a Predictor-Corrector integrator.

This means in particular that the basin of attraction for a certain FP is now independent of the choice of the parameter λ and represents a characteristics of the originally unstable FP and the transformation L_k . In Figure 2 we show a typical ensemble of trajectories which are solutions to eq.(6) for a given set of starting points for the case of the stabilized period two orbit of the Ikeda map (see section 3). Part of the trajectories are running towards one point of the period two orbit and the other part towards the second point of the orbit. The boarderline of the two basins of attraction represents an unstable one dimensional manifold which is the only region of instability concerning the coordinate range of the Ikeda attractor.

We emphasize that the above described method for the detection of the unstable periodic orbits can be applied to any chaotic dynamical system independent of its dimension and individual properties. The advantages of our approach are obvious: given any chaotic dynamical system with a skeleton of unstable periodic orbits we can transform these orbits into stable periodic orbits of a different transformed dissipative dynamical system. A large class of periodic orbits is thereby stabilized by the same linear transformation. The positions of the periodic orbits can then be obtained simply by iterating the transformed system or solving the differential equations of its continuous formulation. In this way the FP can be determined with in principle arbitrary accuracy.

Finally we mention that our method is by no means restricted to discrete dynamical systems. Periodic orbits in continuous dynamical systems can be detected by using the Poincare map which is again a discrete map representing the original continuous dynamical system in a chosen hyperspace. It is hereby not necessary for the Poincare map to be given in an analytical form. The transformed dynamical systems can be obtained from eqs.(2,6) for a numerically given Poincare map. All the quantities given in these equations are defined in the Poincare section which is a (N-1)-dimensional space and periodic orbits represent a finite number of points in this space i.e. are of integer period in the Poincare section. It is therefore not necessary to know the period of the orbits in the continuous system in order to detect them in the Poincare section.

III. APPLICATION TO CHAOTIC DYNAMICAL SYSTEMS

We applied our method to several 2D iterative maps like, for example, the Henon, 2D-logistic and in particular the more complicated Ikeda map. In the following we will discuss in some detail our procedure and the corresponding results for the Ikeda attractor.

To demonstrate the reliability as well as efficiency of our method we use it to calculate the unstable periodic orbits of the Ikeda attractor. The underlying dynamical law is the Ikeda map [15] used in nonlinear optics to describe the response of a 2-level homogeneous absorber in a ring cavity to a constant incident light wave. The map is given by:

$$\begin{aligned} U_I : \quad x_{n+1} &= 1 + 0.9 (x_n \cos w_n - y_n \sin w_n) \\ y_{n+1} &= 0.9 (x_n \sin w_n + y_n \cos w_n) \quad w_n = 0.4 - \frac{6}{1 + x_n^2 + y_n^2} \end{aligned} \quad (7)$$

The Ikeda attractor is fully chaotic and embedded in two dimensional coordinate space. Our considerations for the $N = 2$ case described in section 2 can therefore be applied directly. In detail we proceed as follows: Using the set of stabilization matrices $\mathbf{C}_i, i = 1, \dots, 5$ given in section 2 we construct according to eqs.(2,6) the transformed systems which possess the corresponding stable FP. Starting from an initial point we next produce iteratively a chaotic trajectory on the attractor which provides a set of starting points for the trajectories of the transformed dynamical systems which are solutions to eqs.(2,6). If a certain trajectory converges to a point \vec{r}_F within a given accuracy then this point is recorded as a FP of the Ikeda map. We repeated this procedure for the higher iterates of the Ikeda map up to 13 - th iterate. The number of starting points on the attractor needed to obtain the periodic orbits of period 13 is roughly 5000. This is particularly impressive if we take into account that some of the period 13 orbits differ only from their fourth significant digit on and that the number of FP detected for period 13 is 2522. We are therefore in the position to resolve a large number of close lying periodic orbits. Again we emphasize that we have no rigorous proof for the completeness of the detected orbits but empirical evidence.

The origin of the success of our stabilization transformation is its global character. Thus we need only a very coarse grained lattice of initial conditions to cover the attractor. For the discrete formulation of our transformed dynamical systems (see eq.(2)) the values of the parameter λ necessary to achieve stabilization vary from 10^{-1} for orbits of low period to a few times 10^{-5} for periodic orbits of period 13. Another important feature of our approach is the high accuracy which is controlled by the convergence of the corresponding trajectories. Our calculations yield a relative accuracy of

10^{-13} . Having determined the coordinates of the cycles we use the Ikeda map in eq.(7) to derive the corresponding stability coefficients.

The corresponding results are presented in table I. The first row shows the number of unstable cycles for a given period p and in the second row the total number of cycle points of order N is given. Starting with one period 2, two period 3 and three period 4 cycles their number increases strongly with increasing period and finally for period 12 and 13 we obtain 110 and 194 cycles, respectively. With the help of the obtained orbits and their stability properties we are in the position to determine characteristic quantities of the attractor related to its degree of chaoticity or dynamical and geometrical complexity. To get an impression of the quality of the covering (representation) of the Ikeda attractor through the unstable periodic orbits we illustrate in Fig.3(a) the set of all the FP of the Ikeda map and its higher iterates (up to the 13-th iterate). For comparison we present in Fig.3(b) a typical chaotic trajectory on the attractor which possesses the same total number of points (5627). Although the overall picture looks very similar a more careful comparison of Figs.3(a,b) reveals major differences in the local density of points. If we compare the distribution of the FP shown in Fig.3(a) with the Ikeda attractor of Fig.3(b) we realize that there exist regions within the support of the attractor which are not visited by periodic cycles up to period 13 at all. In addition there exist regions of high density of FP (see Fig.3(a)) which do not correspond to regions of high density for the attractor itself (see Fig.3(b)).

In order to perform a quantitative analysis of the Ikeda attractor in terms of the unstable cycles let us first calculate the measure of the exponential increase (with increasing period) of their number or in other words the so called topological entropy defined through [17]:

$$S_T = \lim_{p \rightarrow \infty} \frac{1}{p} \ln n(p) \quad (8)$$

where $n(p)$ denotes the number of FP of U_I^p . The values of S_T are presented in table I (third row) and they can be used as a rough guide for the completeness of the number of periodic orbits found for a given period. An average measure of the strangeness of the attractor is given in terms of the average Lyapunov exponents defined through [8]:

$$h_{e,c} = \lim_{p \rightarrow \infty} \sum_j \frac{1}{\mu_e(\vec{r}_{jp})} \ln \mu_{e,c}(\vec{r}_{jp}) \quad (9)$$

where \vec{r}_{jp} denotes the j -th FP of the p -th iterate of the map U_I and $\mu_e(\vec{r}_{jp})$ ($\mu_c(\vec{r}_{jp})$) is the expanding (contracting) eigenvalue of the stability matrix at this point. The two exponents correspond to an average expanding rate (h_e) and an average contracting rate (h_c), respectively. The results as a function of period p are presented in the fourth row of Table I. The strong fluctuations in successive terms of this expansion can be seen best in Fig.4(a) where the two Lyapunov exponents are shown as a function of the period p . To characterize the strange attractor geometrically we follow [1] and try to cover the attractor with slabs of length 1 and width $\frac{1}{\mu_c(\vec{r}_{jp})}$. In this case the corresponding Hausdorff dimension D_o can be found by solving the equation:

$$\sum_j \mu_c(\vec{r}_{jp})^{D_o^{(p)}-1} = 1 \quad (10)$$

The numerical results with increasing period p are given in the fifth and last row of table I. The corresponding graphical illustration is shown in Fig.4(b). Again strong oscillations are present even for higher periods.

Summarizing our analysis of the Ikeda attractor we observe a satisfactory convergence of the topological entropy up to period 13 but strongly oscillating values for the fractal dimension as well as average Lyapunov exponents.

IV. CONCLUSIONS

The main objective of the present paper is the development of a general method to detect the unstable periodic orbits in chaotic dynamical systems. The central idea is to convert the unstable fixed points (periodic orbits) to stable ones without changing their location in space. We have shown that this can be achieved by a set of linear transformations, i.e. transformation matrices, which allow the stabilization of any configuration of unstable hyperbolic fixed points for a given dynamical system. The numerical calculation of the stable fixed points in the transformed dissipative dynamical systems is done by either a simple iteration procedure or by solving the corresponding continuous version of the transformed dynamical laws.

The above approach allows a straightforward application to any analytically or numerically given dynamical system. Apart from its general applicability the advantages of our method are the following. It is of universal character in the sense that no previous knowledges about the topological or dynamical behaviour, except the presence of chaos, are required. Our method is by no means restricted to lower dimensional (1D or 2D) systems but can in principle yield periodic orbits for dynamical systems of any dimension. It allows an efficient convergence to highly accurate values and requires at the same time only a small set of initial conditions to cover the strange attractor: a coarse grained covering is sufficient to detect the periodic orbits which are within our empirical procedure assumed to be complete. Nevertheless even close lying periodic orbits of higher periods can be resolved and, therefore, distinguished through the evolution of the stabilized system. The underlying stabilization transformation possesses an appealing geometrical interpretation. While the dynamics of the original chaotic system in the vicinity of the fixed points is characterized by a 'turning' of trajectories [16] in any direction of coordinate space the stabilized system belongs to a vector field which is centered around specified configurations of sinks/sources. The positions of those sinks/sources are identical with the positions of the fixed points. We remark that our method can also be applied to cases where the dynamical system is not fully chaotic but consists of a chaotic sea filled with islands of regularity.

The above method is not restricted to discrete dynamical systems: periodic orbits in continuous dynamical systems can be detected by using the Poincare map which is again a discrete map representing the original continuous dynamical system in a certain hyperspace. As a further perspective and application of the developed scheme we mention the possibility of solving nonlinear equations in general through the iteration of a suitably stabilized version of the original equations. Thereby we translate the problem of finding the roots of a set of nonlinear equations $\vec{f}(\vec{x}) = \vec{0}$ to that of finding the fixed points of the dynamical law $\vec{F}(\vec{x}) = \vec{f}(\vec{x}) + \vec{x}$. Applying the above method to get a stabilized version of \vec{F} we obtain then by simple iteration the fixed points of \vec{F} which correspond to the roots of \vec{f} .

Acknowledgments

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- [1] D. Auerbach, P. Cvitanović, J.-P. Eckmann, G. H. Gunaratne and I. Procaccia, Phys. Rev. Lett. **58**, 2387 (1987)
- [2] P. Cvitanović, Phys. Rev. Lett. **61**, 2729 (1988)
- [3] C. Grebogi, E. Ott, J. A. Yorke, Phys. Rev. **A 37**, 1711 (1988)
- [4] E. Ott, T. Sauer and J. A. Yorke, Phys. Rev. **A 39**, 4212 (1989)
- [5] D. P. Lathrop and A. J. Kostelich, Phys. Rev. **A 40**, 4028 (1989)
- [6] E. Ott, C. Grebogi and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
- [7] M.C. Gutzwiller, “*Chaos in Classical and Quantum Mechanics*”, Springer-Verlag 1990
- [8] E. Ott, “*Chaos in dynamical systems*”, Cambridge University Press (1992), p. 326 and references therein.
- [9] J.P. Eckmann and D. Ruelle, Rev.Mod.Phys.**57**, 617 (1985)
- [10] O. Biham and W. Wenzel, Phys. Rev. Lett. **63**, 819 (1989); Phys. Rev. **A 42**, 4639 (1990)
- [11] P. Grassberger, H. Kantz and U. Moenig, J.Phys.A**22**, 5217 (1989)
- [12] K. Hansen, Phys. Rev. **E 52**, 2388 (1995).
- [13] P. Cvitanović, G. H. Gunaratne and I. Procaccia, Phys. Rev. **A 38**, 1503 (1988)
- [14] P.Schmelcher and F.K.Diakonov, Phys.Rev.Lett.**78**, 4733 (1997)
- [15] K. Ikeda, Opt. Comm. **30**, 257 (1979); S. M. Hammel, C. K. R. T. Jones and J. V. Moloney, J. Opt. Soc. Am. **B 4**, 552 (1985).
- [16] F.K. Diakonov and P. Schmelcher, CHAOS **7**, 239 (1997); acc. f. publ. in Int. J. Bif. and Chaos.
- [17] A. Katok, Pub. Math. IHES **51**, 377 (1980).

Figure Captions

FIG. 1. (a) The vector field \vec{V}_{S_k} belonging to the Ikeda map on its attractor. (b) The corresponding vector field of the second iterate of the Ikeda map around the (stabilized) fixed point (0.5098,-0.6084) belonging to the period two cycle of the Ikeda map. (c) The vector field around the other fixed point (0.6216,0.6059) of the period two cycle. The positions of the fixed points are indicated by crosses. 200 points have been used for each subfigure.

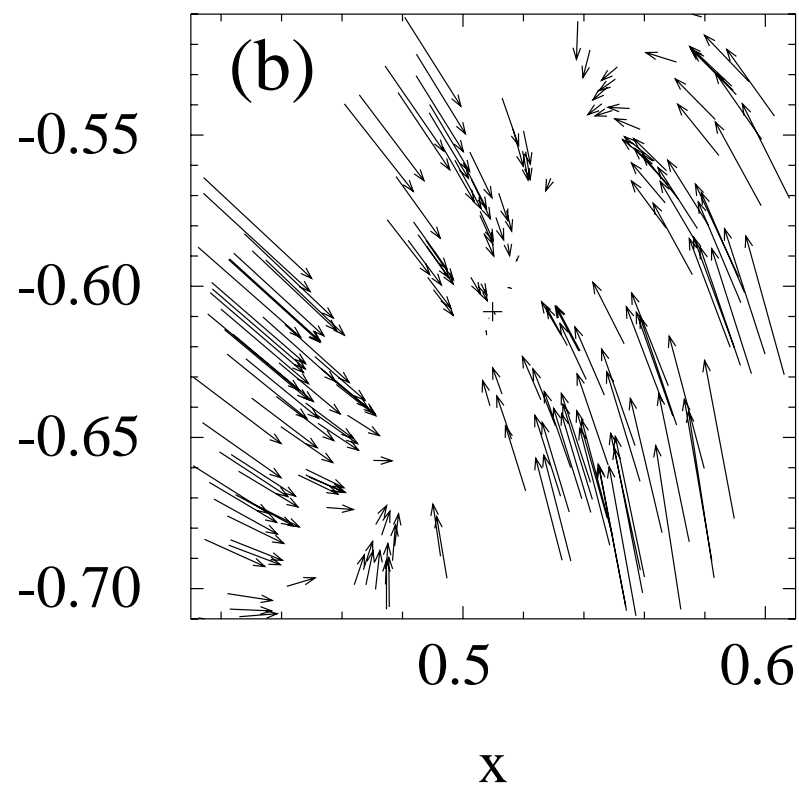
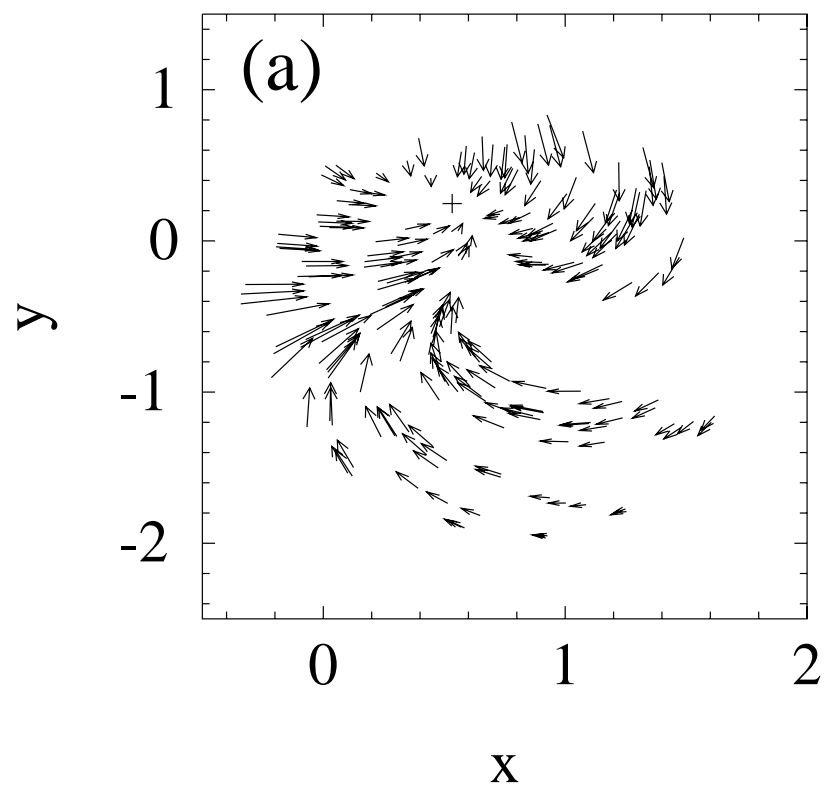
FIG. 2. Lines which represent an ensemble of trajectories of the transformed map in eq.(2) for the stabilization of the period 2 orbit of the Ikeda attractor. The circles indicate the positions of the points of the period 2 orbit and the line connecting the crosses indicates the boarder line between the two basins of attraction.

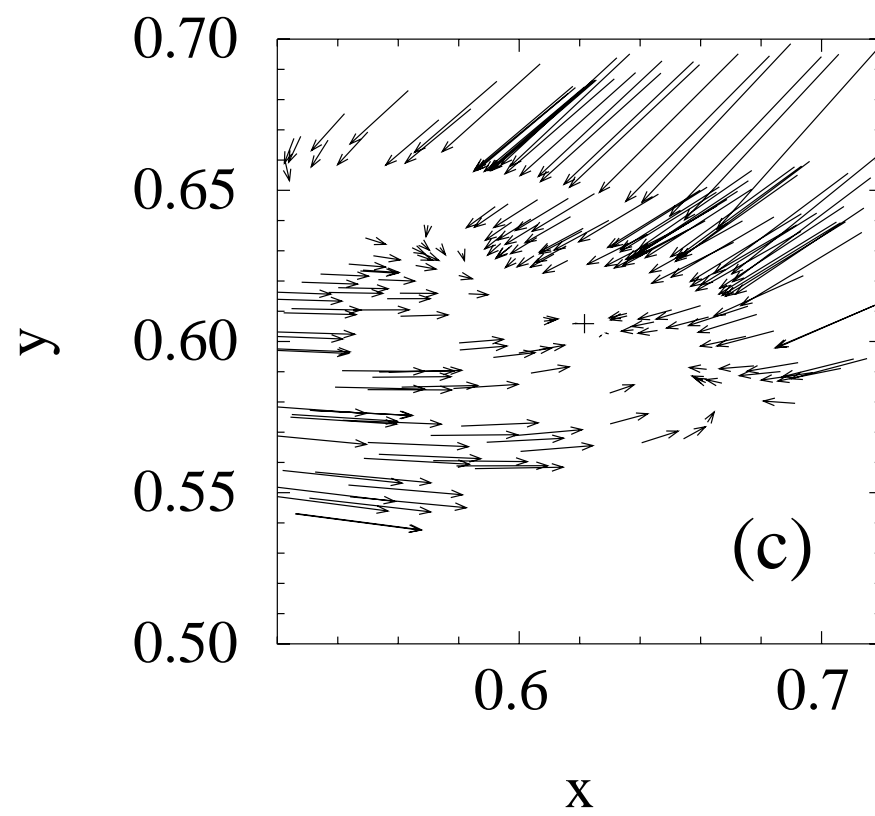
FIG. 3. (a) The set of all the fixed points of the Ikeda map and its higher iterates up to period 13. (b) A typical chaotic trajectory on the Ikeda attractor with the same number of points (5627) as in (a).

FIG. 4. (a) The average Lyapunov exponents as a function of the period p . (b) The fractal dimension D_o of the Ikeda attractor as a function of the period p . See section 3.

Table Captions

Table I: The number of cycles with period N , the total number of cycle points of order N , the topological entropy, the Lyapunov exponents as well as the fractal dimension for $N=1,...,13$ (see section 3).





N	1	2	3	4	5	6	7	8	9	10	11	12	13
Number of cycles with period N	1	1	2	3	4	7	10	14	26	45	76	110	194
Number of cycle points order N	1	3	7	15	21	51	71	127	241	473	837	1383	2523
Topological Entropy	0.0	0.549	0.648	0.677	0.608	0.655	0.609	0.606	0.609	0.616	0.612	0.603	0.603
Lyapunov Exponents	0.365 -0.453	0.513 -0.648	0.547 -0.686	0.735 -0.942	0.597 -0.793	0.884 -1.144	0.490 -0.647	1.071 -1.493	0.854 -1.220	0.957 -1.313	0.688 -0.954	1.046 -1.388	0.468 -0.639
Fractal Dimension	1.00	1.68	1.76	1.95	1.83	2.04	1.78	2.06	1.95	2.02	1.81	2.13	1.76

